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Reduced Electron Models for Edge Simulation

R. H. Cohen*, **M. Dorf,** and **M. Dorr**

Lawrence Livermore National Laboratory, P.O. Box 808, Livermore, CA 94550 USA

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We consider the treatment of electrons in kinetic simulations of edge plasmas. A fully kinetic treatment of electrons is expensive because of the short timescales associated with rapid streaming along field lines. Hence we survey a number of reduced options, of varying complexity, with particular attention to the "adiabatic model" commonly used for core plasmas. We note that the adiabatic model is the linear limit of a Boltzmann model, whose nonlinear form is more appropriate for the large potential variations in edge plasmas, and then consider extensions to the Boltzmann model which render it applicable to a plasma which includes end-loss in the scrape-off layer.

1 Introduction

There is now a number of efforts world-wide to develop whole-edge multi-dimensional kinetic simulation codes. This was in response to the recognition that finite mean-free-path effects along magnetic field lines and finite-ion-radial-orbit widths make the validity of fluid treatments marginal. These new codes are based on drift or gyrokinetics. As with their cousins for core plasmas, the addition of fully kinetic electrons, while desirable, is computationally challenging because of short timescales and difficulties associated with avoiding numerical instabilities. So it is desirable to consider the applicability of reduced models.

2 Survey of simplifying approximations

We begin with the electron drift-kinetic equation (DKE) in total energy $\mathcal{E}$ (including electrostatic potential) and magnetic-moment μ coordinates ,

$$\frac{\partial f}{\partial e} + v_{\|}\frac{\partial f}{\partial s} + \mathbf{v_{dr}}\cdot\nabla f + \dot{\mathcal{E}}\frac{\partial f}{\partial \mathcal{E}} = C(f)\,, \tag{1}$$

where $v_{\|} = [(2/m)(\mathcal{E} - \mu B + e\Phi)]$ and $\dot{\mathcal{E}}$ represents acceleration by any inductive part of the electric field. This equation is a simplification of the full 6-dimensional kinetic equation, valid for phenomena slow compared to the electron gyrofrequency and long in scale compared to the electron gyroradius. It is computationally challenging because the parallel-streaming and collision terms introduce fast timescales into the complete system and also because it introduces fast fluctuation timescales which must be resolved in an explicit code to avoid numerical instability, and even then introduces extraneous noise into simulations. These difficulties can be addressed by introducing implicitness into the parallel-streaming and collision times.

As a further reduction of Eq. (1) one might consider a parallel-stream-averaged (also known as "bounce-averaged") DKE to eliminate the fast parallel-streaming term. This makes sense for high-temperature core simulation, but not in the edge, where collisions are sufficiently strong to compete with parallel streaming over at least parts of the boundary region. There are regimes in which the bounce-averaged DKE is justified, but the limited applicability and computational complexity (non-locality, kinetic phase-space dimensions, and remaining fast timescales) suggest that this equation is not worth the bother.

* Corresponding author. E-mail: rcohen@llnl.gov, Phone: +01 925 422 9831, Fax: +01 925 423 3484

The relatively high electron collisionality leads one to consider electron fluid equations (moments of the DKE, or the high-magnetic-field limit of moments of the full kinetic equation). The full set of electron fluid equations (for density, parallel velocity and temperature) contains fast timescales comparable to the DKE and in that regard provides little benefit except for saving the work of advancing a full kinetic species. However, because the mass appears in the momentum equation and not in the continuity or temperature equations one can consider a limit where the inertia terms are negligible and obtain a useful simplification while preserving electron dynamical terms (and hence, for example, electron-temperature-gradient instabilities).

3 Modified Boltzmann (Adiabatic) Model

The Boltzmann (or, for small perturbations, the "adiabatic electron") model, is just the isothermal limit of the massless electron parallel momentum equation,

$$en_e\frac{\partial\Phi}{\partial s}=T_e\frac{\partial(n_e)}{\partial s} \tag{2}$$

where s denotes distance along a field line and T_e is now considered constant along a field line (but can vary across). Taking the isothermal limit is formally justified only in a low-collisionality limit in which fluid-equations themselves are of questionable validity, so in this regard the Boltzmann-electron model is no more justified than the parallel-averaged DKE, but its simplicity makes it an attractive electron model for kinetic simulation of ion phenomena and is widely used in core simulations. Our objective here is to consider extensions to the common implementation used in the core that give it some degree of physical relevance for edge plasmas.

The immediate consequence of Eq. (2) in the isothermal limit is the Boltzmann form for the electron density variation along a field line,

$$n_e=C\exp(e\Phi/T_e) \tag{3}$$

where C is constant on field lines. There are various possible ways to pick the constant C which correspond to different approximations about electron transport across field lines. A common one is

$$C=\langle n_e(t=0)\rangle/\langle\exp(e\Phi/T_e)\rangle \tag{4}$$

where $\langle\ \rangle$ denotes a field-line average. This choice is consistent with no electron particle transport across field lines, and hence, apart from transients, no ion particle transport either. However, it is consistent with calculation of ion neoclassical thermal and momentum transport. The linearization of Eq. (3) with this choice leads to the commonly used adiabatic electron model,

$$\delta n_e/n_{e0}=e\left(\delta\Phi-\langle\delta\Phi\rangle\right)/T_e\,. \tag{5}$$

For the scrape-off-layer (SOL) region, the presence of open field lines means that the field-line-integrated electron density need not be constant even in the limit of negligible cross-field electron transport. Hence we need a different choice for the pre-Boltzmann constant C, and one is provided by considering current conservation and the endloss-current-regulating property of the sheath in front of the divertor (or limiter) plates. Denoting the current at each end of a field line as positive if it is toward the wall, the current conservation relation integrated along a flux tube can be expressed as [1, 2]

$$\frac{J_{n1}}{\sin\theta_1 B_1}+\frac{J_{n2}}{\sin\theta_2 B_2}+\int ds\frac{\nabla_\perp\cdot\mathbf{J}_\perp}{B}=0 \tag{6}$$

where $\mathbf{J}_\perp$ is the measured (from the simulation) ion radial current density, $\sin\theta$ is the angle between the magnetic field and its projection onto the divertor plate, B_{pj} is the magnetic field strength at end j ($j=1,2$), and the endloss current density at end j, measured normal to the divertor surface, is given by

$$J_{nj}=J_{ni,j}-\sin\theta_j e n_{ej}v_e\exp\left(-e\phi_j/T_e\right) \tag{7}$$

and where $J_{ni,j}$ is the measured (from the simulation) ion endloss current density measured normal to the divertor plate at end j. Here ϕ_j is the difference between the plasma potential Φ and the wall potential (usually, zero)

at end j, and $v_e = (T_e/2\pi m)^{1/2}$. Note that since $J_{ni,j}$ comes from the simulation, it will naturally include the drift currents that contribute to endloss [1–3] as well as loss from ion flow along field lines. Equation (7) coupled with a statement of current conservation has been used numerous times in the past (though usually with a specific approximation for $J_{ni,j}$) in a transversely local (*i.e.* drift-kinetic ions) approximation to determine the potential and net parallel current in the presence of differing plasma conditions at the two ends, biased divertor plates, and/or divertor plates tilted with respect to endwalls; see for examples Refs. [4–7].

Here, we relate the local electron density n_e to the gyro-center ion density N_i including polarization density. In the limit of small ratio of ion gyroradius to potential scale length, this relationship is

$$n_e = N_i + \nabla_\perp \kappa \nabla_\perp \Phi / e \tag{8}$$

where $\kappa = \omega_{pi}^2/4\pi\omega_{ci}^2$. We can now explicitly write the form for the constant (on field lines) C in Eq. (3) for the SOL as

$$C = (N_i + \nabla_\perp \kappa \nabla_\perp \Phi/e) \exp\left(-e\Phi/T_e\right) . \tag{9}$$

where the right-hand side must now be the same evaluated anywhere along a field line within the plasma (excluding the regions within the divertor sheaths). In particular, it can be evaluated on the plasma side of the (gyro)sheath at either of the divertor plates.

Now we close Eq. (6) via the endloss expression (7) using the adiabatic relationship (3) and (9) to obtain the following differential equation:

$$\begin{aligned} &-\left(\frac{\sigma_1}{B_1} + \frac{\sigma_2}{B_2}\right) v_e \left(eN_i + \nabla_\perp \kappa \nabla_\perp \Phi\right) \exp(-e\Phi/T_e) + \\ &\frac{J_{ni,1}}{\sin\theta_1 B_1} + \frac{J_{ni,2}}{\sin\theta_2 B_2} + \int ds \frac{\nabla_\perp \cdot \mathbf{J}_\perp}{B} = 0 \end{aligned} \tag{10}$$

where $\sigma_j = \exp(e\Phi_{wj}/T_e)$ and Φ_{wj} is the wall potential at divertor j ($j = 1$ or 2), so $\sigma_j = 1$ in the usual case of grounded walls. Equation (10) can be interpreted in two ways: It can be interpreted as the full PDE to be solved once boundary conditions have been specified, or it can be viewed, when evaluated at the entrance to the divertor sheaths, as an equation that gives us guidance about boundary conditions.

Focusing first on the latter interpretation, we note that the polarization term is generally small, $\mathcal{O}(\epsilon)$, where ϵ is the square of ratio of the ion gyroradius to L_ϕ, the macroscoppic scale length for variation of Φ, compared to the other terms in the equation. ϵ is $\mathcal{O}(1)$ inside the gyrosheath but is small outside. This estimate is obtained by noting that, from the leading-order solution we discuss below, $\Phi \sim T_e/e$, and, outside of the gyrosheath, its scale length is of order of that for the electron temperature. We obtain similar estimates for either the equilibrium or fluctuation potential solutions; in the latter case, L_ϕ is the inverse of a characteristic wave number; and in any case, the smallness of ϵ is a requirement for validity of the differential form of polarization used in Eq. (8).

Hence to a good approximation Φ at the divertor plates is given by ignoring the polarization term, taking N_i to be the value of the gyro-center ion density adjacent to (but on the plasma side of the gyro sheath at) the divertor plate in question, and evaluating Φ from the resulting algebraic equation. Once Φ along the divertor plates is known from this procedure and an appropriate choice is made for the potential at the radial boundaries, it can be determined everywhere else using (the first interpretation of) Eq. (10). Or, if more convenient, it can be determined, equivalently, by noting the constancy of the left and right hand sides of Eq. (9) along a field line to write:

$$\nabla_\perp \kappa \nabla_\perp \Phi/e = N_{i1} \exp\left(e\left(\Phi - \Phi_1\right)/T_e\right) - N_i \tag{11}$$

where the subscript "1" denotes evaluation on the plasma side of the gyrosheath at divertor plate 1. Here N_{i1}, T_e, and Φ_1 are just known functions of the field-line-label coordinates. (The equation could equally well be written with the subscript "2" everywhere). By construction, the resulting solution to either (10) or (11) will consistently have the polarization term vanishing at the divertor plates.

It should be noted that positing instead a finite polarization at the boundaries (of order ϵ times the ion charge density) would lead to boundary values and a global solution for Φ which differ from the one above by $\mathcal{O}(\epsilon)$.

Hence the prescription above suffices to determine a potential solution with adequate accuracy in the SOL. Compared to the alternative of discarding the polarization term everywhere in the SOL, our prescription has the advantage of providing a solution which matches on with continuous value and slope and with a physically significant second derivative to the potential in the closed-flux-surface edge region. The latter is important for calculating phenomena that depend on the electric-drift velocity shear.

The formulation presented above can be applied to a multi-ion-species plasma with the substitution $N_i \rightarrow \sum Z_i N_i$ and redefining $\kappa = \sum \omega_{pi}^2/(4\pi\omega_{ci}^2)$.

One might be concerned that the solution for Φ obtained from (10) or (11) might be inconsistent with a solution of the gyrokinetic Poisson Equation [Eq. (8) viewed as an equation for Φ] . But in fact the gyrokinetic Poission Equation (8) with the substitution of n_e from (3) and C from (9) is identically satisfied. It can also be verified that we obtain the same potentials at a given physical location if we interchange the identity of the end labels "1" and "2".

4 Modified adiabatic model in the closed-flux-surface region

An issue arises in applying an adiabatic model to a simulation that spans the separatrix: if we apply the standard "core" model [defined by Eq. (4)]. in the closed-flux-surface region, there is (aside from transients) no particle flux in that region, which then implies that, in the absence of local sources, the SOL region must decay to zero density. A sustained SOL requires replenishment via transport from the closed-flux-surface region and/or sources local to the SOL. One way to achieve this is to modify the core model to provide some particle flux. And, one way to achieve that within the framework of the Boltzmann-electron model is to add an (ad-hoc) ambipolar diffusion to both the core electron and ion models. So for example we can add to the ion gyrokinetic equation a term of the form $D\nabla_r^2 f_i$ where ∇_r denotes the gradient with respect to the flux-surface coordinate at constant perpendicular and parallel velocity, f_i is the ion gyro-center distribution, and D is a constant. We then add an auxiliary equation for the flux-surface average of n_e:

$$\frac{\partial \langle n_e \rangle}{\partial t} = D\nabla_r^2 \langle n_e \rangle \tag{12}$$

with $\langle n_e \rangle$ specified on the inner boundary of the simulation. Optionally we can add charge-neutral sources to the equations for f_i and $\langle n_e \rangle$. This prescription is not unique; there are many possible forms for the (possibly velocity-dependent) diffusion term that would yield the same ion particle transport. Here we have picked the simplest possible form. For a more complicated form, Eq. (12) would be modified as needed to ensure ambipolar transport.

5 Summary/Conclusions

We have developed a modified Boltzmann electron model for the closed- and open-flux-surface regions of an edge simulation that are consistent with SOL end-loss, the gyrokinetic-Poisson formulation of the electrostatic potential, and a steadily sustained SOL density profile. This is likely the simplest possible model of electrons for use in kinetic ion simulations that is consistent with basic SOL physics. It should be noted that, to obtain reasonable simulation results from it (in particular, sensible results for the potential solution), the ion simulation model also needs to be sufficiently complete; in particular, it requires sufficient complexity (*e.g.* an ionization source, collisions with neutrals, neoclassical orbits) to seed a density variation along field lines so that a pre-sheath develops.

We would argue that the modified Boltzmann electron model is part of a relatively small heierarchy of reasonable electron models for edge simulation. Specifically, moving up in complexity, one can consider a full set of inertia-less fluid equations, and then the full drift-kinetic equation (DKE). We argue that little else is worth implementation: in particular electron fluid equations with inertia contain the same timescale challenges as the full DKE, while a bounce-averaged DKE is only applicable to hot or tenuous regions within an edge plasma and introduces the additional complication of nonlocality.

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